REFERENCES

1. MOISEYEV N.D., On some general methods for the qualitative study of forms of motion in problems of celestial mechanics: 3. On the construction of regions of continuous stability and continuous instability in Lyapunov's sense. Trudy Gos. Astron. Inst. im. P.K. Shternberga, 9, 2, 1939.
2. ZUBOV V.I., Theory of Oscillations, Vysshaya Shkola, Moscow, 1979.
3. POZHARITSKII G.K., On the construction of the Lyapunov function from integrals of the equations of perturbed motion. Prikl. Mat. Mekh., 22, 2, 1958.
4. CHETAYEV N.G., The Stability of Motion, Gostekhizdat, Moscow, 1955.
5. NEMYTSKII V.V. and STEPANOV V.V., Qualitative Theory of Differential Equations, Gostekhizdat, Moscow-Leningrad, 1949.
6. YAKUBOVICH V.A. and STARZHINSKII V.M., Linear Differential Equations with Periodic Coefficients and their Applications, Nauka, Moscow, 1972.
7. RUBANOVSKII V.N. and STEPANOV S.YA., On Routh's theorem and Chetayev's method for constructing the Lyapunov function from integrals of the equations of motion. Prikl. Mat. Mekh., 33, 5, 1969.
8. SHOSTAK R.YA., On a criterion for the conditional definiteness of a quadratic form in $n$ variables subject to linear constraints, and on a sufficient condition for a conditional extremum of a function of $n$ variables. Uspekhi Mat. Nauk, 9, 2, 1954.
9. ARKHANGEl'SKII YU.A., Analytic Dynamics of Rigid Bodies, Nauka, Moscow, 1977.
10. APPELL P., Theoretical Mechanics, 2, Fizmatgiz, Moscow, 1960.
11. GOLUBEV V.V., Lectures on the intergration of Equations of Motion of a Heavy Rigid Body about a Fixed Point, Gostekhizdat, Moscow, 1953.
12. IRTEGOV V.D., On the stability of pendulum oscillations of the S.V. Kovalevskaya gyroscope. Trudy Kazan. Aviats. Inst., 97, 1968.
13. DOKSHEVICH A.I., Two classes of motions of a Kovalevskaya top. Prikl. Mat. Mekh., 45, 4, 1981.


Translated by D.L.

*PMM U.S.S.R.*, Vol.53, No.6, pp. 695-703, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00


# AN ALGORITHM FOR THE ASYMPTOTIC SOLUTION OF A SINGULARLY PERTURBED LINEAR TIME-OPTIMAL CONTROL PROBLEM*

A.I. KALININ

An algorithm for the approximate solution (in the asymptotic sense) of a singularly perturbed linear time-optimal control problem is proposed. A computational procedure is outlined, which permits the use of the resulting asymptotic approximation for the exact solution of the problem with a prescribed value of the small parameter.

*1. Statement of the problem.* In the class of scalar piecewise-continuous controls, we consider the following optimal control problem for a time-independent linear system:

$$x^{\cdot} = A(\mu)x + b(\mu)u, \quad x(0) = x^\circ, \quad x(T) = 0$$
$$|u(t)| \leqslant 1, \quad J(u) = T \to \min \tag{1.1}$$
$$A(\mu) = \begin{Vmatrix} A_1/\mu & A_2/\mu \\ A_3 & A_4 \end{Vmatrix}, \quad b(\mu) = \begin{Vmatrix} b_1/\mu \\ b_2 \end{Vmatrix}, \quad x = \begin{Vmatrix} z \\ y \end{Vmatrix}, \quad x^\circ = \begin{Vmatrix} z^\circ \\ y^\circ \end{Vmatrix}$$

*Prikl.Matem.Mekhan., 53, 6, 880-889, 1989

where $\mu$ is a small positive parameter, $z$ is an $n$-vector, $y$ is an $m$-vector; the other elements of the problem have the appropriate dimensions. The following conditions are assumed to hold:

a) the matrix $A_1$ is stable, i.e., the real parts of all its eigenvalues are negative.

b) $\operatorname{rank}(b_1, A_1b_1, \ldots, A_1^{n-1}b_1) = n$.

Problem (1.1) and its generalizations have been the subject of many publications (e.g., /1-4/). Most of the studies published to date are qualitative in nature. In particular, it has been shown that as $\mu \to 0$ the terminal time $T^\circ(\mu)$ in problem (1.1) tends to the terminal time $T^\circ$ in the problem

$$\begin{gathered} y^\cdot = A_0y + b_0u, \quad y(0) = y^\circ, \quad y(T) = 0 \\ |u(t)| \leqslant 1, \quad J_0(u) = T \to \min \\ A_0 = A_4 - A_3A_1^{-1}A_2, \quad b_0 = b_2 - A_3A_1^{-1}b_1 \end{gathered} \tag{1.2}$$

As to optimal control switching points in the singularly perturbed problem, some of them are close to the corresponding switching points in problem (1.2), while the others lag behind the terminal time $T^\circ(\mu)$ by an amount of the order of $\mu$. In some cases additional switching points may appear, concentrated in the vicinity of the initial time.

*Definition.* A piecewise-continuous control $u(t, \mu)$, $t \in [0, T(\mu)]$ satisfying the constraint $|u(t, \mu)| \leqslant 1$, $t \in [0, T(\mu)]$ is said to be asymptotically $N$-optimal in problem (1.1) if the trajectory $z(t, \mu)$, $y(t, \mu)$, $t \in [0, T(\mu)]$ which it generates satisfies the conditions $z(T(\mu), \mu) = O_1(\mu^{N+1})$, $y(T(\mu), \mu) = O_2(\mu^{N+1})$, and $T(\mu) - T^\circ(\mu) = O_3(\mu^{N+1})$.

In this paper we propose an algorithm which, given a natural number $N$, constructs an asymptotically $N$-optimal control for the problem in question. Essentially, the algorithm determines the asymptotic behaviour of optimal control switching points and the time $T^\circ(\mu)$. The computational procedure is based on the direct support method of /5/ for solving linear optimal control problems and on the boundary-function method of /6/. In addition, we shall show how to use the asymptotic approximations produced by the algorithm to obtain an exact solution of problem (1.1) for prescribed values of the small parameter.

2. ***First basic problem.*** The first block of the algorithm solves problem (1.2), which we shall call the "first basic problem". We shall assume that

c) problem (1.2) has a solution and is "simple" /7/.

The solution is obtained using the direct support method of /5/. After a finite number of iterations of the direct and adjoint systems, we obtain:

1) the optimal time $T^\circ$;

2) an optimal control and a trajectory, $u^\circ(t)$, $y^\circ(t)$, $t \in [0, T^\circ]$;

3) a support $\{\tau_1^\circ, \ldots, \tau_{m-1}^\circ\}$, i.e., a set of $m-1$ distinct points in the interval $]0, T^\circ[$ such that the $(m \times (m-1))$ matrix

$$\Phi_0 = (\varphi_0(\tau_j^\circ), \quad j = 1, 2, \ldots, m-1) \tag{2.1}$$

is of full rank, where

$$\varphi_0(t) = F_0(t)\, b_0, \quad t \in [0, T^\circ] \tag{2.2}$$

and $F_0(t)$, $t \in [0, T^\circ]$, is an $(m \times m)$ matrix-valued function satisfying the equation

$$F_0^\cdot = -F_0A_0, \quad F_0(T^\circ) = E \tag{2.3}$$

4) an $m$-vector $\lambda^\circ$, which is a non-trivial solution of the system of homogeneous linear algebraic equations $\Phi_0'\lambda = 0$;

5) a cocontrol $\Delta_0(t) = \psi^{\circ\prime}(t)\, b_0$, $t \in [0, T^\circ]$, derived from the solution $\psi^\circ(t)$, $t \in [0, T^\circ]$, of the adjoint system $\psi^{\cdot\circ} = -A_0'\psi^\circ$, $\psi^\circ(T^\circ) = \lambda^\circ$. We observe that

$$\Delta_0(t) = \lambda^{\circ\prime}\varphi_0(t), \quad t \in [0, T^\circ] \tag{2.4}$$

The cocontrol is related to the optimal control by the equation $u^\circ(t) = -\operatorname{sgn}\Delta_0(t)$, $t \in [0, T^\circ]$, and it has the following property: $\Delta_0(\tau_j^\circ) = 0$, $\Delta_0^\cdot(\tau_j^\circ) \neq 0$, $j = 1, 2, \ldots, m-1$. Let $t_1^\circ, \ldots, t_l^\circ$ denote all the zeros of the cocontrol, indexed in increasing order. Since the sequence of zeros includes the support times, it follows that $l \geqslant m-1$. We shall also assume that

d) $t_j^\circ \in\, ]0, T^\circ[$, $\Delta_0^\cdot(t_j^\circ) \neq 0$, $j = 1, 2, \ldots, l$.

3. ***Second basic problem.*** The second stage of the algorithm solves the following variable-length optimal control problem:

$$\begin{gathered} dz/ds = A_1z + b_1u, \quad s \leqslant 0, \quad z(s_1) = A_1^{-1}b_1, \quad z(0) = 0 \\ |u(s)| \leqslant 1, \quad J_1(u) = \int_{s_1}^{0} (u(s) + 1)\, ds \to \min \end{gathered} \tag{3.1}$$

If conditions (a) and (b) are satisfied, this problem, which we shall refer to as the "second basic problem", has admissible controls.

We shall assume that

e) problem (3.1) has a solution.

Note that the point $A_1^{-1}b_1$ is the equilibrium position of a dynamic system under the control $u(s) \equiv -1$. Therefore, in order to determine an optimal control for the second basic problem, it will suffice to solve the following fixed-length optimal control problem:

$$dz/ds = A_1 z + b_1 u, \quad s \leqslant 0, \quad z(s^*) = A_1^{-1} b_1, \quad z(0) = 0 \tag{3.2}$$

$$|u(s)| \leqslant 1, \quad J_*(u) = \int_{s^*}^{0} u(s)\, ds \to \min$$

where $s^*$ is a sufficiently small negative number. If $s_1^\circ$ is an optimal initial time in problem (3.1), then the optimal control in problem (3.2), considered over the interval $[s_1^\circ, 0]$, is also an optimal control for the second basic problem, and if $s < s_1^\circ$ then $u^*(s) \equiv -1$.

We shall assume that

f) problem (3.2) is "simple".

Solving it by the direct support method, we obtain

1) an optimal control and trajectory $u^*(s)$, $z^*(s)$, $s \in [s^*, 0]$;

2) a support $\{\sigma_1^\circ, \ldots, \sigma_n^\circ\}$, i.e., a sequence of $n$ distinct points in the interval $]s^*, 0[$, such that the $n \times n$ matrix

$$\Pi\Phi = (\Pi\varphi(\sigma_i^\circ), \quad i = 1, 2, \ldots, n) \tag{3.3}$$

called the support matrix, is non-singular, where

$$\Pi\varphi(s) = G(s)\, b_1 \tag{3.4}$$

and $G(s)$, $s \leqslant 0$, is an $n \times n$ matrix-valued function satisfying the equation

$$dG/ds = -GA_1, \quad G(0) = E \tag{3.5}$$

3) a vector of potentials $\pi$, which is a solution of the following system of linear algebraic equations: $\pi' \Pi\varphi(\sigma_i^\circ) = -1$, $i = 1, 2, \ldots, n$;

4) a cocontrol $\Pi\Delta(s) = \Pi\psi'(s)\, b_1 + 1$, $s \in [s^*, 0]$, where $\Pi\psi(s)$, $s \leqslant 0$, is a solution of the adjoint system

$$d\Pi\psi(s)/ds = -A_1' \Pi\psi(s), \quad \Pi\psi(0) = \pi$$

We observe that

$$\Pi\Delta(s) = \pi' \Pi\varphi(s) + 1 \tag{3.6}$$

The cocontrol is related to the optimal control by the equation $u^*(s) = -\operatorname{sgn} \Pi\Delta(s)$, $s \in [s^*, 0]$, and it has the following property:

$$\Pi\Delta(\sigma_i^\circ) = 0, \quad d\Pi\Delta(\sigma_i^\circ)/ds \neq 0, \quad i = 1, 2, \ldots, n$$

Let $s_1^\circ, \ldots, s_p^\circ$ denote all the zeros of the cocontrol, indexed in increasing order. Obviously, $p \geqslant n$. We shall assume that

g) $s_p^\circ \neq 0$, $d\Pi\Delta(s_i^\circ)/ds \neq 0$, $i = 1, 2, \ldots, p$.

If $s^*$ is chosen to be fairly small, then $s_1^\circ$ is an optimal initial time and $s_2^\circ, \ldots, s_p^\circ$ are optimal control switching times in problem (3.1), $z^*(s_1^\circ) = A_1^{-1} b_1$, $u^*(s) = -1$ for $s < s_1^\circ$. The function $\Pi\Delta(s)$, $s \leqslant 0$, will have no zeros other than $s_1^\circ, \ldots, s_p^\circ$, and moreover $\Pi\Delta(s) > 0$ for $s < s_1^\circ$.

After solving the basic problems we find the vector

$$v^\circ = \Delta_0(T^\circ)\, \pi - (A_3 A_1^{-1})' \lambda^\circ = \lambda^{\circ\prime} b_0 \pi - (A_3 A_1^{-1})' \lambda^\circ \tag{3.7}$$

The vector $\lambda^\circ$ is determined uniquely apart from a positive factor. We shall assume that $\| v^\circ \|^2 + \| \lambda^\circ \|^2 = 1$.

***4. The main theorem.*** Our subsequent calculations are based on the following assertions.

*Theorem.* If conditions (a)-(g) are satisfied and $\mu$ is sufficiently small, then problem (1.1) has an optimal control expressible as

$$u^\circ(t, \mu) = \begin{cases} \operatorname{sgn} \Delta_0^{\cdot}(t_1^\circ), & t \in [0, t_1[ \\ \operatorname{sgn} \Delta_0^{\cdot}(t_j^\circ), & t \in [t_{j-1}, t_j[, \quad j = 2, 3, \ldots, l \\ -\operatorname{sgn} \Delta_0^{\cdot}(t_l^\circ), & t \in [t_l, \ T + \mu s_1[ \\ (-1)^i \operatorname{sgn} \Delta_0^{\cdot}(t_l^\circ), & t \in [T + \mu s_{i-1}, T + \mu s_i[, \quad i = 2, 3, \ldots, p \\ (-1)^{p+1} \operatorname{sgn} \Delta_0^{\cdot}(t_l^\circ), & t \in [T + \mu s_p, T] \end{cases} \tag{4.1}$$

where the functions

$$T = T(\mu);\quad t_j = t_j(\mu),\quad j = 1, 2, \ldots, l;\quad s_i = s_i(\mu),\quad i = 1, 2, \ldots, p \tag{4.2}$$

have the asymptotic expansions

$$T \sim \sum \mu^k T^k,\quad t_j \sim \sum \mu^k t_j^k,\quad s_i \sim \sum \mu^k s_i^k \tag{4.3}$$

Throughout, the symbol $\Sigma$ denotes summation from $k = 0$ to $k = \infty$.

Let $\psi(t, \mu)$, $t \in [0, T(\mu)]$, be the vector of conjugate variables corresponding to Eq.(4.1) by the maximum principle /8/,

$$\mu\nu = -(\psi_j(T(\mu), \mu),\ j = 1, 2, \ldots, n)',\quad \lambda = -(\psi_{n+i}(T(\mu), \mu),\ i = 1, 2, \ldots, m)'$$

where $\|\nu\|^2 + \|\lambda\|^2 = 1$. Then the vector-functions $\nu(\mu), \lambda(\mu)$ have asymptotic expansions

$$\nu \sim \sum \mu^k \nu^k,\quad \lambda \sim \sum \mu^k \lambda^k \tag{4.4}$$

and, together with the functions (4.2), they solve the system of equations

$$\begin{gathered} x(T, t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu) = 0 \\ \psi'(t_j, \nu, \lambda, T, \mu)\, b(\mu) = 0,\quad j = 1, 2, \ldots, l \\ \psi'(T + \mu s_i, \nu, \lambda, T, \mu)\, b(\mu) = 0,\quad i = 1, 2, \ldots, p \\ (\|\nu\|^2 + \|\lambda\|^2)/2 - 1/2 = 0 \end{gathered} \tag{4.5}$$

where $x(t, t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu)$, $t \in [0, T]$, is the trajectory of the singularly perturbed system generated by the initial state $x(0) = x^0$ and the control $u(t, t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu)$, $t \in [0, T]$, of type (4.1); $\psi(t, \nu, \lambda, T, \mu)$, $t \in [0, T]$ is a solution of the adjoint system

$$\psi^{\cdot} = -A'(\mu)\psi,\quad \psi(T) = \begin{Vmatrix} \mu\nu \\ \lambda \end{Vmatrix} \tag{4.6}$$

*Proof*. Using Cauchy's formula to represent the solution of the singularly perturbed system generated by the control $u(t, t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu)$, $t \in [0, T]$, we obtain

$$\begin{gathered} x(T, t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu) = F(0, T, \mu)x^0 + \\ \operatorname{sgn}\Delta_0^{\cdot}(t_1^\circ)\int_0^{t_1}\varphi\, dt + \ldots + \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_{l-1}}^{t_l}\varphi\, dt - \\ \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\Big[\int_{t_l}^{T+\mu s_1}\varphi\, dt - \ldots + (-1)^p\int_{T+\mu s_p}^{T}\varphi\, dt\Big] \end{gathered} \tag{4.7}$$

$$\varphi = F(t, T, \mu)\, b(\mu) \tag{4.8}$$

Here $F(t, T, \mu)$, $t \in [0, T]$, is an $(n + m) \times (n + m)$ matrix-valued function, which is a solution of the singularly perturbed equation

$$F^{\cdot} = -FA(\mu),\quad F(T) = E \tag{4.9}$$

and exhibits the following block structure:

$$F = \begin{Vmatrix} F_1 & F_2 \\ F_3 & F_4 \end{Vmatrix}$$

where $F_i = F_i(t, T, \mu)$, $t \in [0, T]$, $i = 1, 2, 3, 4$, are matrices of orders $n \times n$, $n \times m$, $m \times n$, $m \times m$, respectively. Using the boundary-function method of /6/, one can expand these matrices in asymptotic series

$$F_i \sim \sum \mu^k [F_{ik}(t, T) + \Pi_k F_i(s)],\quad s = (t - T)/\mu,\quad t \in [0, T],\quad i = 1, 2, 3, 4 \tag{4.10}$$

We emphasize that these are uniform asymptotic expansions. It is also essential here that the functions $\Pi_k F_i(s)$, $s \leqslant 0$, called the boundary terms, satisfy the estimates

$$\|\Pi_k F_i(s)\| \leqslant \alpha_k \exp(\beta_k s), \quad i = 1, 2, 3, 4, \quad k = 0, 1, \ldots \tag{4.11}$$

where $\alpha_k, \beta_k$ are certain positive constants.

We will specify a few of the first terms of the expansions (4.10):

$$\begin{gathered} F_{10} = 0, \quad F_{20} = -A_1^{-1}A_2F_0(t, T), \quad F_{30} = 0, \quad F_{40} = F_0(t, T) \\ F_{11} = A_1^{-1}A_2F_0(t, T)A_3A_1^{-1}, \quad F_{31} = -F_0(t, T)A_3A_1^{-1} \\ \Pi_0F_1 = G(s), \quad \Pi_0F_2 = G(s)A_1^{-1}A_2, \quad \Pi_0F_3 = 0 \\ \Pi_0F_4 = 0, \quad \Pi_1F_3 = A_3A_1^{-1}G(s) \end{gathered} \tag{4.12}$$

where $F_0(t, T)$, $t \in [0, T]$, is an $m \times m$ matrix-valued function, which is a solution of the equation

$$F_0^{\cdot} = -F_0A_0, \quad F_0(T) = E \tag{4.13}$$

and $G(s)$, $s \leqslant 0$, satisfies (3.5).

Let $\varphi_1(t, T, \mu)$, $\varphi_2(t, T, \mu)$, $t \in [0, T]$, be vector-valued functions whose components are respectively the first $n$ and last $m$ components of $\varphi(t, T, \mu)$. Then, as is evident from (3.4), (4.8), (4.10) and (4.12), we have the following uniform asymptotic expansions:

$$\varphi_1 \sim \Pi\varphi(s)/\mu + \sum \mu^k [\varphi_{1k}(t, T) + \Pi_k\varphi_1(s)] \tag{4.14}$$

$$\begin{gathered} \varphi_2 \sim \sum \mu^k [\varphi_{2k}(t, T) + \Pi_k\varphi_2(s)] \\ \varphi_{1k} = F_{1,k+1}b_1 + F_{2k}b_2, \quad \Pi_k\varphi_1 = \Pi_{k+1}F_1b_1 + \Pi_kF_2b_2 \\ \varphi_{2k} = F_{3,k+1}b_1 + F_{4k}b_2, \quad \Pi_k\varphi_2 = \Pi_{k+1}F_3b_1 + \Pi_kF_4b_2 \end{gathered} \tag{4.15}$$

Note that by (2.2), (2.3), (3.4), (4.12) and (4.13),

$$\begin{gathered} \varphi_{10}(t, T^\circ) = -A_1^{-1}A_2\varphi_0(t), \quad \varphi_{20}(t, T^\circ) = \varphi_0(t), \quad t \in [0, T^\circ] \\ \Pi_0\varphi_2(s) = A_3A_1^{-1}\Pi\varphi(s), \quad s \leqslant 0 \end{gathered} \tag{4.16}$$

Let $z(t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu)$, $y(t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu)$ be vector-valued functions whose components are respectively the first $n$ and last $m$ components of (4.7). As follows from (4.10)-(4.12), (4.14), (4.15),

$$\begin{gathered} z \sim \sum \mu^k z_k(t_1, \ldots, t_l, s_1, \ldots, s_p, T) \\ y \sim \sum \mu^k y_k(t_1, \ldots, t_l, s_1, \ldots, s_p, T) \end{gathered} \tag{4.17}$$

$$\begin{gathered} z_0 = -A_1^{-1}A_2F_0(0, T)y^\circ + \operatorname{sgn}\Delta_0^{\cdot}(t_1^\circ)\int_0^{t_1}\varphi_{10}\,dt + \ldots \\ + \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_{l-1}}^{t_l}\varphi_{10}\,dt - \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_l}^{T}\varphi_{10}\,dt + \\ \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\Big[-\int_{-\infty}^{s_1}\Pi\varphi\,ds + \int_{s_1}^{s_2}\Pi\varphi\,ds - \ldots + (-1)^{p+1}\int_{s_p}^{0}\Pi\varphi\,ds \end{gathered} \tag{4.18}$$

$$y_0 = F_0(0, T)y^\circ + \operatorname{sgn}\Delta_0^{\cdot}(t_1^\circ)\int_0^{t_1}\varphi_{20}\,dt + \ldots + \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_{l-1}}^{t_l}\varphi_{20}\,dt - \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_l}^{T}\varphi_{20}\,dt \tag{4.19}$$

$$\begin{gathered} z_k = F_{1k}(0, T)z^\circ + F_{2k}(0, T)y^\circ + \operatorname{sgn}\Delta_0^{\cdot}(t_1^\circ)\int_0^{t_1}\varphi_{1k}\,dt + \ldots \\ + \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_{l-1}}^{t_l}\varphi_{1k}\,dt - \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\int_{t_l}^{T}\varphi_{1k}\,dt + \\ \operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\Big[-\int_{-\infty}^{s_1}\Pi_{k-1}\varphi_1\,ds + \int_{s_1}^{s_2}\Pi_{k-1}\varphi_1\,ds - \ldots \\ + (-1)^{p+1}\int_{s_p}^{0}\Pi_{k-1}\varphi_1\,ds\Big] - 2\operatorname{sgn}\Delta_0^{\cdot}(t_l^\circ)\sum_{i=1}^{k}\frac{1}{i!}\frac{\partial^{i-1}}{\partial t^{i-1}}\varphi_{1,k-i}(T, T)\times \\ [(s_1)^i - (s_2)^i + \ldots + (-1)^{p+1}(s_p)^i], \quad k = 1, 2, \ldots \end{gathered}$$

A similar formula is valid for $y_k$, $k \geqslant 1$, with the sole difference that $F_{1k}$ is replaced by $F_{3k}$, $F_{2k}$ by $F_{4k}$, and $\varphi_{1k}$, $\Pi_{k-1}\varphi_1$ by $\varphi_{2k}$, $\Pi_{k-1}\varphi_2$, respectively.

Put $\Delta(t, \nu, \lambda, T, \mu) = \psi'(t, \nu, \lambda, T, \mu) b(\mu)$, $t \in [0, T]$. As is evident from (4.6), (4.8), (4.9), $\Delta = \mu\nu'\varphi_1 + \lambda'\varphi_2$. But then, by (4.14), this function has a uniform asymptotic expansion

$$\Delta \sim \sum \mu^k [\Delta_k(t, \nu, \lambda, T) + \Pi_k \Delta(s, \nu, \lambda)] \tag{4.20}$$
$$\Delta_0 = \lambda'\varphi_{20},\ \Delta_k = \nu'\varphi_{1,k-1} + \lambda'\varphi_{2k},\ k = 1, 2, \ldots$$
$$\Pi_0\Delta = \nu'\Pi\varphi + \lambda'\Pi_0\varphi_2,\ \Pi_k\Delta = \nu'\Pi_{k-1}\varphi_1 + \lambda'\Pi_k\varphi_2,\ k = 1, 2, \ldots$$

Note that (2.4), (3.6), (3.7) and (4.16) imply

$$\Delta_0(t, \nu^\circ, \lambda^\circ, T^\circ) = \Delta_0(t),\quad t \in [0, T^\circ] \tag{4.21}$$
$$\Pi_0\Delta(s, \nu^\circ, \lambda^\circ) = \Delta_0(T^\circ)(\Pi\Delta(s) - 1),\ s \leqslant 0$$

Let $\delta(s, \nu, \lambda, T, \mu) = \Delta(T + \mu s, \nu, \lambda, T, \mu)$, $s \leqslant 0$. By (4.20) and the fact that $t = T + \mu s$, we have

$$\delta \sim \sum \mu^k \delta_k(s, \nu, \lambda, T) \tag{4.22}$$
$$\delta_k = \Pi_k\Delta(s, \nu, \lambda) + \sum_{i=0}^{k} \frac{s^i}{i!} \frac{\partial^i}{\partial t^i} \Delta_{k-i}(T, \nu, \lambda, T) \tag{4.23}$$

Let $h = (t_1, \ldots, t_l, s_1, \ldots, s_p, T, \nu_1, \ldots, \nu_n, \lambda_1, \ldots, \lambda_m)'$, where $\nu_i$, $i = 1, 2, \ldots, n$, are the components of $\nu$ and $\lambda_j$, $j = 1, 2, \ldots, m$, those of $\lambda$. Then system (4.5) may be written as

$$R(h, \mu) = 0 \tag{4.24}$$

$$R(h, \mu) = \left\| \begin{array}{l} z(t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu) \\ y(t_1, \ldots, t_l, s_1, \ldots, s_p, T, \mu) \\ \Delta(t_j, \nu, \lambda, T, \mu),\quad j = 1, 2, \ldots, l \\ \delta(s_i, \nu, \lambda, T, \mu),\quad i = 1, 2, \ldots, p \\ (\|\nu\|^2 + \|\lambda\|^2)/2 - {}^1/_2 \end{array} \right\|$$

As follows from (4.17), (4.20), (4.22) and the estimates (4.11), the left-hand side of Eq.(4.24) may be expanded asymptotically as

$$R(h, \mu) \sim \Sigma \mu^k R_k(h) \tag{4.25}$$

$$R_k(h) = \left\| \begin{array}{l} z_k(t_1, \ldots, t_l, s_1, \ldots, s_p, T) \\ y_k(t_1, \ldots, t_l, s_1, \ldots, s_p, T) \\ \Delta_k(t_j, \nu, \lambda, T),\quad j = 1, 2, \ldots, l \\ \delta_k(s_i, \nu, \lambda, T),\quad i = 1, 2, \ldots, p \\ r_k(\nu, \lambda) \end{array} \right\| \tag{4.26}$$
$$r_0 = (\|\nu\|^2 + \|\lambda\|^2)/2 - 1/2,\quad r_k = 0,\quad k \geqslant 1$$

Define $R(h, 0) = R_0(h)$. Then the vector-valued function $R(h, \mu)$ is continuous in the domain $\|h - h_0\| < \eta_0$, $0 \leqslant \mu < \mu_0$, together with its partial derivatives with respect to the components of $h$. Here $\eta_0$, $\mu_0$ are certain small positive numbers and

$$h_0 = (t_1^\circ, \ldots, t_l^\circ, s_1^\circ, \ldots, s_p^\circ, T^\circ, \nu_1^\circ, \ldots, \nu_n^\circ, \lambda_1^\circ, \ldots, \lambda_m^\circ)'.$$

Relying on the fact that the controls $u^\circ(t)$, $t \in [0, T^\circ]$, $u^*(s)$, $s \in [s_1^\circ, 0]$, are admissible in problems (1.2), (3.1) and also on (4.16), (4.18), (4.19), (4.21), (4.23), (4.26), one can show that $R(h_0, 0) = R_0(h_0) = 0$.

It can be shown by direct differentiation that the Jacobian of system (4.24) has the following structure:

$$I_0 = \left\| \begin{array}{ccccc} B_1 & B_2 & c_1 & 0 & 0 \\ B_3 & 0 & c_2 & 0 & 0 \\ B_4 & 0 & c_3 & 0 & B_5 \\ 0 & B_6 & 0 & B_7 & B_8 \\ 0 & 0 & 0 & \nu^{\circ\prime} & \lambda^{\circ\prime} \end{array} \right\| \tag{4.27}$$

$$B_1 = (-2A_1^{-1}A_2\varphi_0(t_j^\circ)\,\mathrm{sgn}\,\Delta_0^\cdot(t_j^\circ),\ j = 1, 2, \ldots, l)$$
$$B_2 = (2(-1)^{f-l}\Pi\varphi(s_i^\circ)\,\mathrm{sgn}\,\Delta_0^\cdot(t_l^\circ),\ i = 1, 2, \ldots, p)$$
$$B_3 = (2\varphi_0(t_j^\circ)\,\mathrm{sgn}\,\Delta_0^\cdot(t_j^\circ),\ j = 1, 2, \ldots, l)$$
$$B_4 = \mathrm{diag}\,(\Delta_0^\cdot(t_j^\circ), j = 1, 2, \ldots, l),\ B_5 = (\varphi_0(t_j^\circ), j = 1, 2, \ldots, l)'$$
$$B_6 = \mathrm{diag}\,(\Delta_0(T^\circ)d\Pi\Delta(s_i^\circ)/ds,\ i = 1, 2, \ldots, p)$$
$$B_7 = (\Pi\varphi(s_i^\circ),\ i = 1, 2, \ldots, p)',\ B_8 = (A_3A_1^{-1}\Pi\varphi(s_i^\circ) + b_0,$$
$$i = 1, 2, \ldots, p)'$$
$$c_1 = A_1^{-1}A_2b_0\,\mathrm{sgn}\,\Delta_0^\cdot(t_l^\circ),\quad c_2 = -b_0\,\mathrm{sgn}\,\Delta_0^\cdot(t_l^\circ)$$
$$c_3 = (\lambda^{\circ\prime}A_0\varphi_0(t_j^\circ),\ j = 1, 2, \ldots, l)'$$

Using the fact that the matrices (2.1), (3.3) have full rank, one can show that the rank of the matrix obtained from (4.27) by deleting the last row and the $l + p + 1$-th column is $l + p + n + m - 1$. Since

$$\lambda^{\circ\prime}\varphi_0(t_j^\circ) = \Delta_0(t_j^\circ) = 0,\ j = 1, 2, \ldots, l;\ \nu^{\circ\prime}\Pi\varphi(s_i^\circ) +$$
$$\lambda^{\circ\prime}A_3A_1^{-1}\Pi\varphi(s_i^\circ) + \lambda^{\circ\prime}b_0 = \Delta_0(T^\circ)\Pi\Delta(s_i^\circ) = 0,\quad i = 1, 2, \ldots, p$$

all the rows of this matrix are orthogonal to the vector $(0, 0, \nu^{\circ\prime}, \lambda^{\circ\prime})$. Hence it follows that the columns of $I_0$, with the exception of the $(l + p + 1)$-th, are linearly independent. Moreover, they are orthogonal to the vector $(0, \lambda^{\circ\prime}, 0, 0, 0)$. But the $l + p + 1$-th column is not orthogonal to this vector, since $\lambda^{\circ\prime}b_0 = \Delta_0(T^\circ) \neq 0$, implying that the Jacobian is non-singular.

Thus, system(4.24) or, what is the same,(4.5) satisfies all the conditions of the Implicit Function Theorem. This means that for sufficiently small $\mu$ problem (1.1) has an admissible control $u^\circ(t, \mu)$, $t \in [0, T(\mu)]$, of the type (4.1), and there exist vectors $\nu(\mu)$, $\lambda(\mu)$ such that the switching points of $u^\circ(t, \mu)$ are the zeros of the function $\Delta(t, \mu) = \psi'(t, \mu)\,b(\mu)$, $t \in [0, T(\mu)]$, where $\psi'(t, \mu)$, $t \in [0, T(\mu)]$, is a non-trivial solution of the adjoint system (4.6) with $T = T(\mu)$, $\nu = \nu(\mu)$, $\lambda = \lambda(\mu)$.

Since the left-hand sides of system (4.5) can be expanded in asymptotic series in integer powers of $\mu$, it follows /9/ that the asymptotic expansions (4.3), (4.4) exist.

Note that $\Delta(t, \mu) = \Delta(t, \nu(\mu), \lambda(\mu), T(\mu), \mu)$, $t \in [0, T(\mu)]$. Hence it follows from (4.20), (4.21) that there exists a constant $C > 0$ for which

$$|\Delta(t, \mu) - \Delta_0(t) - \Delta_0(T^\circ)(\Pi\Delta(s) - 1)| \leqslant C\mu$$
$$s = (t - T(\mu))/\mu,\quad t \in [0, T(\mu)]$$

Relying on this fact, assumptions (d) and (g) and the remarks in Sect.3 about the function $\Pi\Delta(s)$, $s \leqslant 0$, one can show that for sufficiently small $\mu$ the cocontrol $\Delta(t, \mu)$, $t \in [0, T(\mu)]$, has no zeros other than the switching points of the control $u^\circ(t, \mu)$, $t \in [0, T(\mu)]$, where $u^\circ(t, \mu) = -\,\mathrm{sgn}\,\Delta(t, \mu)$, $t \in [0, T(\mu)]$. But this means that for sufficiently small $\mu$ the admissible control $u^\circ(t, \mu)$, $t \in [0, T(\mu)]$ satisfies the Pontryagin Maximum Principle /8/, and so it is an optimal control. This completes the proof of the theorem.

5. ***Construction of asymptotic expansions.*** A control of type (4.1) with $t_j = t_j^\circ$, $j = 1, 2, \ldots, l$; $s_i = s_i^\circ$, $i = 1, 2, \ldots, p$; $T = T^\circ$ is an asymptotically 0-optimal control for problem (1.1). To contruct an $N$-optimal control $(N \geqslant 1)$, it suffices to find the coefficients

$$t_j^k,\ j = 1, 2, \ldots, l;\ s_i^k,\ j = 1, 2, \ldots, p;\ T^k,\ k = 1, 2, \ldots, N \tag{5.1}$$

of the expansions (4.3). Let

$$h_k = (t_1^k, \ldots, t_l^k, s_1^k, \ldots, s_p^k, T^k, \nu_1^k, \ldots, \nu_n^k, \lambda_1^k, \ldots, \lambda_m^k)'$$
$$h_N(\mu) = \textstyle\sum_N \mu^k h_k$$

Throughout, $\Sigma_N$ denotes summation from $k = 0$ to $k = N$. Expand the vector-valued function $\Sigma_N\mu^kR_k(h_N(\mu))$ in powers of $\mu$ up to order $N$ inclusive and equate the expansion coefficients to zero. This gives non-singular systems of linear equations for successive determination of the vectors $h_k$, $k = 1, 2, \ldots, N$:

$$I_0h_1 = -R_1(h_0) \tag{5.2}$$
$$I_0h_2 = -\frac{\partial R_1}{\partial h}(h_0)\,h_1 - \frac{1}{2}h_1'\frac{\partial^2R_0}{\partial h^2}(h_0)\,h_1 - R_2(h_0)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We note that thanks to the structure of the Jacobian $I_0$ (see (4.27)) each of systems (5.2) splits: we first use a system of order $n+m+1$ to determine the coefficients $T^k, \nu^k, \lambda^k$ and then, independently, determine the remaining coefficients $t_j^k$, $j=1,2,\ldots,l$; $s_i^k$, $i=1,2,\ldots,p$. If $l=m-1, p=n$, i.e., if the cocontrols of the basic problems have no zeros that are not support points, the initial system (4.5) splits: the optimal time and switching times (4.1) can be found independently of the Lagrange multipliers. This naturally implies a corresponding decomposition of systems (5.2).

Successively solving systems (5.2), we find the coefficients (5.1) and construct polynomials

$$T^N(\mu) = \sum_N \mu^k T^k, \quad t_j^N(\mu) = \sum_N \mu^k t_j^k, \quad j=1,2,\ldots,l$$
$$s_i^N(\mu) = \sum_N \mu^k s_i^k, \quad i=1,2,\ldots,p$$

The control (4.1), where $t_j = t_j^N(\mu)$, $j=1,2,\ldots,l$; $s_i = s_i^N(\mu)$, $i=1,2,\ldots,p$; $T=T^N(\mu)$, is an asymptotically $N$-optimal control for problem (1.1).

The above asymptotic approximations to the roots of Eq.(4.24) can be used for the exact solution of the equation and hence of the problem as a whole, for a prescribed value of the small parameter. To that end one uses the "updating" procedure of /5/, i.e, Newton's method, to find the roots of Eq.(4.24), taking $h_N(\mu)$ as the initial approximation. When this is done the matrix $\partial R(h,\mu)/\partial h$ can be replaced by its asymptotic expansion, whose coefficients are determined from those of the expansion (4.25).

We might mention in conclusion that there are no essential difficulties in devising an analogous algorithm for multidimensional control systems.

**6. *Example.*** Consider the following example, which describes the control of a DC motor /2/:

$$\mu z^{\cdot} = -z - \frac{k}{k_1} y + b_1 u, \quad z(0) = z^\circ, \quad z(T) = 0 \tag{6.1}$$
$$y^{\cdot} = \frac{k_1}{1+k} z - \frac{1}{1+k} y, \quad y(0) = y^\circ, \quad y(T) = 0$$
$$|u| \leqslant 1, \quad J(u) = T \to \min$$

All constants in problem (6.1) are positive. Applying our algorithm, we find an asymptotically 0-optimal control:

$$u(t,\mu) = \begin{cases} -1, & t \in [0,\ T^\circ + \mu s^\circ[ \\ 1, & t \in [T^\circ + \mu s^\circ,\ T^\circ] \end{cases}$$

and a 1-optimal control:

$$u^1(t,\mu) = \begin{cases} -1, & t \in [0,\ T^\circ + \mu(T^1 + s^\circ) + \mu^2 s^1[ \\ 1, & t \in [T^\circ + \mu(T^1 + s^\circ) + \mu^2 s^1,\ T^\circ + \mu T^1] \end{cases}$$
$$T^\circ = \ln(1 + y^\circ/b_0), \quad b_0 = k_1 b_1/(1+k), \quad s^\circ = -\ln 2$$
$$T^1 = -y_1/b_0, \quad s^1 = -k y_1/k_1 b_1 - z_1/b_1$$
$$z_1 = \frac{kb_1}{1+k}(1 + 3s^\circ - \exp(-T^\circ)) - \frac{k^2 b_1}{(1+k)^2}(1 - \exp(-T^\circ) + T^\circ \exp(-T^\circ)) +$$
$$\frac{k(2k+1)b_1}{(1+k)^2}(1 - \exp(-T^\circ)) - \frac{k}{1+k}\exp(-T^\circ)\left(z^\circ + \frac{2k+1}{k_1} y^\circ - \frac{kT^\circ y^\circ}{k_1}\right)$$
$$y_1 = \frac{\exp(-T^\circ)}{1+k}(k_1 z^\circ + k y^\circ - k y^\circ T^\circ) - \frac{kb_0}{1+k}(1 - \exp(-T^\circ) + T^\circ \exp(-T^\circ)) +$$
$$b_0(2 - 2s^\circ - \exp(-T^\circ))$$

## REFERENCES

1. COLLINS W.D., Singular perturbations of linear time-optimal control problems. In: Recent Mathematical Developments in Control. Academic Press, London, New York, 1973.
2. KOKOTOVIC P.V. and HADDAD A.H., Controllability and time-optimal control of systems with slow and fast models. IEEE Trans. Automat. Control, 20, 1, 1975.
3. KOKOTOVIC P.V. and HADDAD A.H., Singular perturbations of a class of time-optimal controls. IEEE Trans. Automat. Control, 20, 1, 1975.
4. GICHEV T.R. and DONCHEV A.L., Convergence of the solution of a linear singularly perturbed time-optimal problem. Prikl. Mat. Mekh., 43, 4, 1979.
5. GABASOV R. and KIRILLOVA F.M., Constructive Methods of Optimization, II: Control Problems, Universitetskoye, Minsk, 1984.

6. VASIL'YEVA A.B. and BUTUZOV V.F., Asymptotic Expansions of Solutions of Singularly Perturbed Equations, Nauka, Moscow, 1973.
7. GABASOV R., KIRILLOVA F.M. and KOSTYUKOVA O.I., Optimization of control sytems using multiple supports. In: Constructive Theory of Extremal Problems, Universitetskoye, Minsk, 1984.
8. PONTRYAGIN L.S., BOLTYANSKII V.G., GAMKRELIDZE R.V. and MISHCHENKO E.F., Mathematical Theory of Optimal Processes, Nauka, Moscow, 1983.
9. VAINBERG M.M. and TRENOGIN V.A., Branching Theory of Solutions of Non-linear Equations, Nauka, Moscow, 1969.

Translated by D.L.

*PMM U.S.S.R.*, Vol.53, No.6, pp.703-707, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00


# STABILIZATION OF WEAKLY LINEAR SYSTEMS*

V.A. KOLOSOV

The problem of stabilizing bilinear systems, characterized by the presence of a small parameter in the bilinear part of the system, is considered. The result is an approximate method for synthesizing a stabilizing control /1-3/ in bilinear systems, in the case of a performance index. Estimates are derived for the error with respect to the performance index.

1. ***Statement of the problem.*** Suppose we are given a bilinear control system

$$x^{\cdot} = \varepsilon N(t)xu + B(t)u;\ x \in R_n;\quad x(0) = x_0;\ t \geqslant 0 \tag{1.1}$$

Here $N(t)$ is a measurable and bounded $n \times n$ matrix for $t \geqslant 0$; $B(t) \in R_n$ is a vector-valued function, also measurable and bounded for $t \geqslant 0$. The problem is to determine a scalar control in the class $U$ of bounded controls $u = u(t, x)$, $\varepsilon \geqslant 0$ is a small parameter.

We wish to synthesize an optimal control in class $U$, which stabilizes system (1.1). The performance index is

$$J(u) = \int (x'Q(t)x + \lambda(t)^{-1}u^2)\,dt \tag{1.2}$$

Here $Q(t)$ is a continuous, bounded, uniformly positive definite $n \times n$ matrix, and $\lambda(t)$ is a positive definite scalar function; the prime denotes transposition. Integration with respect to $t$ is always from 0 to $\infty$.

2. ***Successive approximations algorithm.*** Let us assume that for the values of $\varepsilon$ under consideration problem (1.1), (1.2) has a solution. Bellman's equation is

$$\inf_{u \in U} [\partial V/\partial t + u(B(t) + \varepsilon N(t)x)'\partial V/\partial x + x'Q(t)x + \lambda(t)^{-1}u^2] = 0, \tag{2.1}$$
$$(V = V(t, x))$$

It follows from (2.1) that the following expression defines an optimal control:

$$u_*(t, x) = -\frac{1}{2}\lambda(t)(B(t) + \varepsilon N(t)x)'\,\partial V/\partial x \tag{2.2}$$

Expand the function $V$ in powers of $\varepsilon$:

$$V = V_0(t, x) + \varepsilon V_1(t, x) + \ldots \tag{2.3}$$

*Prikl.Matem.Mekhan.*, 53, 6, 890-894, 1989